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TITLE: Monotonicity Conditions of Curvature for Bezier-de Casteljau Curves

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TITLE: International Conference on Curves and Surfaces [4th], Saint-Malo, France, 1-7 July 1999. Proceedings, Volume 1. Curve and Surface Design

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Monotonicity Conditions of Curvature for Bézier-de Casteljau Curves

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Abstract. In this paper, we deal with the monotonicity of curvature problem for Bézier-de Casteljau curves. We focus more particularly on the cubic case. A condition about decreasing curvature at the origin of a cubic curve is given so that it implies decreasing curvature at every point. The corresponding cubics are determined by their control polygon.

§1. Introduction

The problem discussed here concerns the shape control of curves; mainly how to obtain curves with a monotone variation of curvature. The aim is to find the widest class of curves with monotonely increasing or decreasing curvature variation. This problem mainly arises in car body shape design.

Let P be a *n-degree* Bézier-de Casteljau planar curve defined on $[0,1]$ given by its control polygon $\{P_0, P_1, \dots, P_n\}$. Let us consider $l = \mid P_0P_1 \mid$, and, $\forall i = 1, \dots, n-1$, set $h_i = \mid P_iP_{i+1} \mid / \mid P_{i-1}P_i \mid$ and $\varphi_i = (P_iP_{i+1}, P_{i-1}P_i)$. We say that P admits the representation $(h_1, \dots, h_{n-1}; \varphi_1, \dots, \varphi_{n-1}; l)$. Higashi, Kaneko and Hosaka [3] characterized the monotonicity of curvature when $\forall k = 1, \dots, n-1$, $h_k = h$ and $\varphi_k = \varphi$: $h \cos\varphi \geq 1$ and $h \leq \cos\varphi$ are respectively the condition of decrease and increase of the curvature. This model has been used recently by Mineur, Lychah, Castelain and Giaume [4] to control a shape when fitting a curve to a set of given data.

Here, we focus on the cubic case. From the representation $(h_1, h_2; \varphi_1, \varphi_2; l)$ of P, we apply the de Casteljau Algorithm at t value belonging to the interval of definition, and we determine by induction the parameters characterizing the two segments given by the Subdivision Algorithm. This process enables us to determine the curvature ρ. Then, we seek cubic curves for which decreasing curvature at the origin implies decreasing curvature at every point. Such curves are determined *via* the parameters $r^* = h_2/h_1$, φ_1 and φ_2. The study falls into two cases : $r^* \geq 1$ and $r^* < 1$. For the second case, we give a more

Curve and Surface Design: Saint-Malo 1999
Pierre-Jean Laurent, Paul Sablonnière, and Larry L. Schumaker (eds.), pp. 99–108.

ISBN 0-8265-1356-5.

strict decreasing curvature condition at the origin in order to get the decrease of ρ everywhere.

In the framework of Fiorot and Jeannin [1] and Fiorot, Jeannin and Taleb [2], an attempt to extend these results to rational cubic curves has been made.

§2. de Casteljau Algorithm

Let P be a Bézier-de Casteljau cubic curve (BCc in short) defined on $[0,1]$ with its control polygon $\{P_0,P_1,P_2,P_3\}$ and representation $(h_1,h_2;\varphi_1,\varphi_2;l)$. For $t \in [0,1]$, the de Casteljau algorithm gives the points $(P_k^{(j)})_{k=0,\cdots,3-j}^{j=0,\cdots,3}$ defined by the relation $P_k^{(j+1)} = (1-t)P_k^{(j)} + tP_{k+1}^{(j)}$ and we obtain $P(t) = P_0^{(3)}$. Now, we set $\forall k = 1,2$,

$$\gamma_k^{(1)} = \left(P_{k-1}^{(1)}P_{k-1}^{(2)}, P_{k-1}P_{k-1}^{(1)}\right), \qquad \delta_k^{(1)} = \left(P_k^{(1)}P_{k+1}, P_{k-1}^{(2)}P_k^{(1)}\right),$$

$$\Delta_k^{(0)} = \frac{|\, P_{k-1}^{(1)}P_{k-1}^{(2)} \,|}{|\, P_{k-1}P_{k-1}^{(1)} \,|},$$

and

$$h_1^{(1)} = \frac{|\, P_1^{(1)}P_2^{(1)} \,|}{|\, P_0^{(1)}P_1^{(1)} \,|}, \quad \varphi_1^{(1)} = \left(P_1^{(1)}P_2^{(1)}, P_0^{(1)}P_1^{(1)}\right), \quad \Delta_1^{(1)} = \frac{|\, P_0^{(2)}P_0^{(3)} \,|}{|\, P_0^{(1)}P_0^{(2)} \,|},$$

$$\gamma_1^{(2)} = \left(P_0^{(2)}P_0^{(3)}, P_0^{(1)}P_0^{(2)}\right), \qquad \delta_1^{(2)} = \left(P_1^{(2)}P_2^{(1)}, P_0^{(3)}P_1^{(2)}\right).$$

After some calculations, we obtain $\forall k = 1,2$,

$$\Delta_k^{(0)} = \left((1-t)^2 + 2t(1-t)h_k\cos\varphi_k + (h_kt)^2\right)^{\frac{1}{2}}. \tag{1}$$

Then,

$$\exp(i\gamma_k^{(1)}) = \left((1-t) + h_k\exp(i\varphi_k)t\right)/\Delta_k^{(0)}, \tag{2}$$

$$\exp(i\delta_k^{(1)}) = \left((1-t)\exp(i\varphi_k) + h_kt\right)/\Delta_k^{(0)}, \tag{3}$$

$$\varphi_1^{(1)} = \gamma_2^{(1)} + \delta_1^{(1)} \quad [2\pi], \tag{4}$$

$$h_1^{(1)} = \frac{\Delta_2^{(0)}}{\Delta_1^{(0)}}h_1, \tag{5}$$

$$\Delta_1^{(1)} = \left((1-t)^2 + 2t(1-t)h_1^{(1)}\cos\varphi_1^{(1)} + (h_1^{(1)}t)^2\right)^{\frac{1}{2}}, \tag{6}$$

$$\exp(i\gamma_1^{(2)}) = \left((1-t) + h_1^{(1)}\exp(i\varphi_1^{(1)})t\right)/\Delta_1^{(1)}, \tag{7}$$

$$\exp(i\delta_1^{(2)}) = \left((1-t)\exp(i\varphi_1^{(1)}) + h_1^{(1)}t\right)/\Delta_1^{(1)}. \tag{8}$$

Remark 1. For $t \in [0,1]$, the de Casteljau Algorithm splits the curve P into two segments: the left one is the restriction of P to the interval $[0,t]$ defined by the control polygon $\{P_0\ ,\ P_0^{(1)},\ P_0^{(2)},\ P_0^{(3)}\}$ which admits the representation $(\Delta_1^{(0)}, \Delta_1^{(1)}; \gamma_1^{(1)}, \gamma_1^{(2)}; lt)$, whereas the right one is the restriction to $[t,1]$ defined by $\{P_0^{(3)},\ P_1^{(2)},\ P_2^{(1)},\ P_3\}$ which admits $(h_1^{(1)}/\Delta_1^{(1)}, h_2/\Delta_2^{(0)};\ \delta_1^{(2)}, \delta_2^{(1)};\ \Delta_1^{(0)}\Delta_1^{(1)}l(1-t))$ as a representation.

§3. Curvature Characterization

We determine the curvature radius R and the curvature $\rho = \frac{1}{|R|}$ at every point of the curve P *via* the parameters mentioned above. Let us remember that the curvature radius R is determined by the approximation $R \simeq \frac{\Delta s}{\Delta\alpha}$, where s denotes the arc length and α the angle of the tangent vector with a fixed direction.

First, we give the expression of R at $t = 0$. Let us consider a value t near 0. We have the following approximations

$$\Delta s \simeq (1 + \Delta_1^{(0)} + \Delta_1^{(0)}\Delta_1^{(1)})lt,$$
$$\Delta\alpha = -\gamma_1^{(1)} - \gamma_1^{(2)} \simeq -\sin(\gamma_1^{(1)} + \gamma_1^{(2)}).$$

By using (1) – (7), we obtain when t tends to 0

$$R(0) = -\frac{3}{2}\frac{l}{h_1 \sin\varphi_1}. \tag{9}$$

Now for $t \in (0,1]$, we apply (9) to the right segment whose origin is $P_0^{(3)}$. Then, the curvature radius at t is

$$R(t) = -\frac{3}{2}\frac{\Delta_1^{(0)}\Delta_1^{(1)}l(1-t)}{(h_1^{(1)}/\Delta_1^{(1)})\sin\delta_1^{(2)}}.$$

Therefore, (8) implies

$$R(t) = -\frac{3}{2}\frac{\Delta_1^{(0)}(\Delta_1^{(1)})^3 l}{h_1^{(1)}\sin\varphi_1^{(1)}}. \tag{10}$$

Application. Let us consider the case $h_1 = h_2 = h$ and $\varphi_1 = \varphi_2 = \varphi$ with $\varphi \in [0, \frac{\pi}{2}]$. One can prove that $\forall t \in [0,1]$, $\Delta_1^{(0)} = \Delta_1^{(1)} (= \Delta)$, $\varphi_1^{(1)} = \varphi$ and $h_1^{(1)} = h$. Then, we obtain with (10), $\forall t \in [0,1]$, $\rho(t) = 2h\sin\varphi/(3l\Delta^4)$. Differentiating $\Delta = \left((1-t)^2 + 2t(1-t)h\cos\varphi + (ht)^2\right)^{\frac{1}{2}}$, we deduce that $h\cos\varphi \geq 1$ and $h \leq \cos\varphi$ are respectively the condition of decrease and increase of the curvature [3].

§4. Decreasing Curvature Condition in the Case $r^* \geq 1$

Let P be a BCc curve with representation $(h_1, h_2; \varphi_1, \varphi_2; l)$, and let $(\varphi_1, \varphi_2) \in [0, \frac{\pi}{2})^2$. For $t \in [0,1]$, we set $r = \Delta_1^{(1)}/\Delta_1^{(0)}$. At $t = 1$, this parameter is r^*. Moreover, we define

$$\lambda = \frac{\sin(\gamma_1^{(1)} + \gamma_1^{(2)})}{\sin 2\gamma_1^{(1)}}, \qquad \mu = \frac{\sin(\gamma_1^{(1)} + \gamma_1^{(2)})}{\sin 2\gamma_1^{(2)}}.$$

At $t = 1$, we prove *via* (1) – (7) that these parameters are respectively

$$\lambda^* = \frac{\sin(\varphi_1 + \varphi_2)}{\sin 2\varphi_1}, \qquad \mu^* = \frac{\sin(\varphi_1 + \varphi_2)}{\sin 2\varphi_2}.$$

Lemma 1. *At $t = 0$ and $t = 1$, we have respectively the equivalences :*

$$\frac{3}{2} h_1 \cos\varphi_1 \left(1 - \frac{r^*\lambda^*}{3}\right) \geq 1 \Longleftrightarrow \rho'(0) \leq 0, \tag{11}$$

$$h_2 \geq \frac{3}{2}\left(1 - \frac{r^*\mu^*}{3}\right)\cos\varphi_2 \Longleftrightarrow \rho'(1) \leq 0. \tag{12}$$

Proof: We calculate $\rho'(0)$ as $\lim_{t\to 0}(\rho(t) - \rho(0))/t$ *via* (9) and (10). We obtain

$$\rho'(0) = \rho(0)(h_1(h_2 \sin(\varphi_1 + \varphi_2) - 2\sin\varphi_1 - 6\sin\varphi_1(h_1\cos\varphi_1 - 1)). \tag{13}$$

Then, with the above definitions of λ^* and μ^*, we obtain (11). At $t = 1$, we consider the curve $P(1-t)$ which has the representation $(1/h_2, 1/h_1; \varphi_2, \varphi_1; h_1 h_2 l)$. □

Remark 2. The equivalences (11) and (12) do not depend on the interval of definition.

Lemma 2. *Let P be a BCc curve defined on interval $[t_1, t_2]$ with the representation $(h_1, h_2; \varphi_1, \varphi_2; l)$. Let us suppose that $r^* \geq 1$. Then,*

$$\rho'(t_1) \leq 0 \Rightarrow \rho'(t_2) \leq 0.$$

Proof: We have the successive inequalities

$$h_2 \geq h_1 \geq \frac{2}{3}\left(1 - \frac{r^*\lambda^*}{3}\right)^{-1} \frac{1}{\cos\varphi_1} \geq \frac{3}{2}\left(1 - \frac{r^*\mu^*}{3}\right)\cos\varphi_2.$$

The first one uses the definition of r^*. The second one uses (11). From the identity

$$\sin^2(\varphi_1 + \varphi_2) - \sin^2(\varphi_1 - \varphi_2) = \sin 2\varphi_1 \sin 2\varphi_2,$$

we deduce $\lambda^*\mu^* \geq 1$. After some calculations, we obtain the third one and consequently (12). □

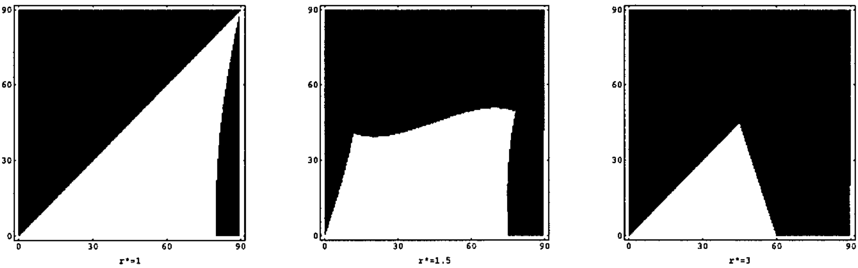

Fig. 1. Decreasing curvature domains for $r^* = 1, 1.5$ *and* 3.

Lemma 3. *Let P be a BCc curve defined on* $[0,1]$. *With the previous notation and hypothesis, we have*

$$r^* \cos(\varphi_1 + \varphi_2) - \cos 2\varphi_1 \geq 0 \Leftrightarrow \forall t \in [0,1], \quad r \geq 1.$$

Proof: For $t \in [0,1]$, we find that $(\Delta_1^{(0)} \Delta_1^{(1)})^2 - (\Delta_1^{(0)})^4$ is positive iff

$$(1-t)^2 A_0 + 2t(1-t)A_1 + t^2 A_2 \geq 0$$

with

$$\begin{cases} A_0 = 2(r^* \cos(\varphi_1 + \varphi_2) - \cos 2\varphi_1), \\ A_1 = 2h_1(r^* \cos\varphi_2 - \cos\varphi_1), \\ A_2 = h_1^2(r^{*2} - 1), \end{cases}$$

is positive. Then $r^* \cos(\varphi_1 + \varphi_2) - \cos 2\varphi_1 \geq 0$. Conversely, this inequality is equivalent to

$$(r^* \cos\varphi_2 - \cos\varphi_1)\cos\varphi_1 \geq (r^* \sin\varphi_2 - \sin\varphi_1)\sin\varphi_1.$$

Considering the cases $\varphi_1 \leq \varphi_2$ and $\varphi_2 \leq \varphi_1$, we see that the coefficients of the above Bernstein polynomial are positive. □

Proposition 1. *Let P be a BCc curve defined on* $[0,1]$ *with the representation* $(h_1, h_2; \varphi_1, \varphi_2; l)$. *Let us suppose that* $r^* \geq 1$. *We consider the domain*

$$D_1 = \{(\varphi_1, \varphi_2) \in [0, \frac{\pi}{2}), r^* \cos(\varphi_1 + \varphi_2) - \cos 2\varphi_1 \geq 0\}.$$

For $(\varphi_1, \varphi_2) \in D_1$, *we have*

$$\rho'(0) \leq 0 \Leftrightarrow \forall t \in [0,1], \quad \rho'(t) \leq 0.$$

Proof: The proof is a consequence of Lemmas 1 and 2. □

The domains corresponding to different values of r^* are described *via* the following graphs in function of φ_1 (horizontal axis) and φ_2 (vertical axis). In Figure 1, the white part represents the decreasing curvature domain whereas the dark one denotes a domain where the decrease at $t = 0$ is not possible ($r^* \lambda^* > 3$). One can notice that the latter is empty when $r^* > 6$. In the grey part, we cannot say anything about the monotonicity.

§5. Decreasing Curvature Condition in the Case $r^* < 1$

With $r^* < 1$, the condition $\rho'(0) \le 0$ is too strict, so we consider the following sufficient condition on decreasing curvature at the origin:

$$\rho'(0) \le 4\left(1 - \frac{1}{r^*}\right)\rho(0),$$

which is equivalent to

$$\frac{3}{2}\, h_1 \cos\varphi_1 \left(1 - \frac{r^*\lambda^*}{3}\right) \ge r^{*-1}. \tag{14}$$

If we set

$$\alpha^* = \frac{3}{2}\, h_1 \cos\varphi_1 \left(1 - \frac{r^*\lambda^*}{3}\right),$$

(14) becomes $r^*\alpha^* \ge 1$. Furthermore, we set $\forall t \in [0,1]$,

$$\alpha = \frac{3}{2}\, \Delta_1^{(0)} \cos\gamma_1^{(1)} \left(1 - \frac{r\lambda}{3}\right).$$

A calculation gives $\alpha = (1-t) + \alpha^* t$.

Remark 3 . The inequality (14) does not depend on the interval of definition.

Lemma 4. *Let P be a BCc curve defined on interval $[t_1, t_2]$ with the representation $(h_1, h_2; \varphi_1, \varphi_2; l)$ and $r^* < 1$. Let us consider domain*

$$D_2 = \{(\varphi_1, \varphi_2) \in [0, \frac{\pi}{2}[, r^* \cos\varphi_2 - \cos\varphi_1 \ge 0\}.$$

Then $\forall\ (\varphi_1, \varphi_2) \in D_2$,

$$\rho'(t_1) \le 4\left(1 - \frac{1}{r^*}\right)\rho(t_1) \Rightarrow \rho'(t_2) \le 0.$$

Proof: For $(\varphi_1, \varphi_2) \in D_2$, we have

$$\left(1 - \frac{r^*\lambda^*}{3}\right)\left(1 - \frac{r^*\mu^*}{3}\right)\cos\varphi_1 \cos\varphi_2 \le \left(\frac{2}{3}\right)^2,$$

which, with (14), implies

$$h_2 = r^* h_1 \ge \frac{2}{3}\left(1 - \frac{r^*\lambda^*}{3}\right)^{-1} \frac{1}{\cos\varphi_1} \ge \frac{3}{2}\left(1 - \frac{r^*\mu^*}{3}\right)\cos\varphi_2. \qquad \square$$

Lemma 5. *Let P be a BCc curve defined on* $[0,1]$. *Under the previous hypothesis, we have*

$$\forall(\varphi_1,\varphi_2)\in D_2 \Leftrightarrow \forall t\in[0,1],\quad r\cos\gamma_1^{(2)}-\cos\gamma_1^{(1)}\geq 0.$$

Proof: For $t\in[0,1]$, we find that $r\cos\gamma_1^{(2)}-\cos\gamma_1^{(1)}\geq 0$ if and only if

$$(1-t)(r^*\cos(\varphi_1+\varphi_2)-\cos 2\varphi_1)+h_1(r^*\cos\varphi_2-\cos\varphi_1)t\geq 0.$$

Then $r^*\cos\varphi_2-\cos\varphi_1\geq 0$. Conversely, $r^*\cos\varphi_2-\cos\varphi_1\geq 0$ and $r^*<1$ imply $\varphi_2\leq\varphi_1$. Then $r^*\sin\varphi_2-\sin\varphi_1\leq 0$. Consequently,

$$r^*\cos(\varphi_1{+}\varphi_2){-}\cos 2\varphi_1{=}(r^*\cos(\varphi_1{+}\varphi_2){-}\cos 2\varphi_1)\cos\varphi_1{-}(r^*\sin\varphi_2{-}\sin\varphi_1)\sin\varphi_1$$

is positive. □

Lemma 6. *Let P be a BCc curve defined on* $[0,1]$. *Under the previous hypothesis, we have* $\forall(\varphi_1,\varphi_2)\in D_2$,

$$h_1({r^*}^2\alpha^*-1)+2(r^*{\alpha^*}^2\cos\varphi_2-\cos\varphi_1)\geq 0\Rightarrow\forall t\in[0,1],\quad r\alpha\geq 1.$$

Proof: The last inequality is equivalent to deciding whether a *fifth-degree* polynomial is positive. We verify that all its coefficients are positive but for one. The positivity of this coefficient is equivalent to the first inequality in the lemma. □

Proposition 2. *Let P be a BCc curve defined on* $[0,1]$ *with the representation* $(h_1,h_2;\varphi_1,\varphi_2;l)$. *Let us suppose that* $r^*<1$. *We consider the domain D_2 as mentioned above and*

$$D_3=\{(\varphi_1,\varphi_2)\in[0,\frac{\pi}{2}),h_1({r^*}^2\alpha^*-1)+2(r^*{\alpha^*}^2\cos\varphi_2-\cos\varphi_1)\geq 0\}.$$

Then, $\forall(\varphi_1,\varphi_2)\in D_2\cap D_3$,

$$\rho'(0)\leq 4\left(1-\frac{1}{r^*}\right)\rho(0)\Rightarrow\forall t\in[0,1],\quad\rho'(t)\leq 0.$$

Proof: The proof is a consequence of Lemmas 5,6 and 4. □

Here, we describe the different admissibility domains $D_2\cap D_3$ (represented in white) for several values of r^* in considering that $\alpha^*=1/r^*$. As illustrated in Figure 2, there is a domain continuity when r^* is near 1 with the case $r^*=1$ (Figure 1). When r^* decreases, the domain gets smaller and smaller and then it is finally empty for $r^*\simeq 0,33$. If we take $\alpha^*\gg 1/r^*$, the domain size increases and tends to D_2.

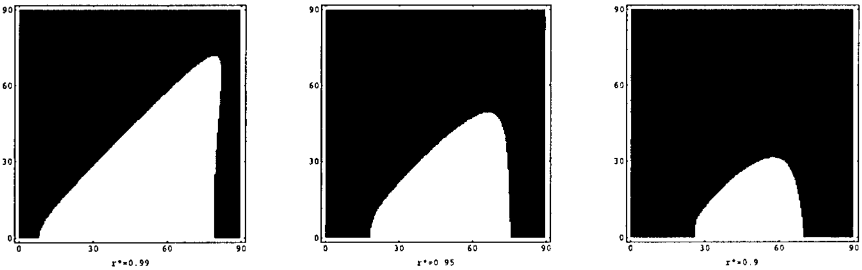

Fig. 2. Decreasing curvature domains for $r^* = 0.99, 0.95, 0.9$ *and* $\alpha^* = 1/r^*$.

§6. Examples

Our results are illustrated by examples of *BCc* curves whose curvature decreases. For each curve, one quarter of the curvature radius is represented at t values 0, 0.1, 0.2, ..., 1.

Example 1. $r^* = 1$ and $\rho'(0) = 0$. The first curve with small angles $(\frac{\pi}{7}, \frac{\pi}{16})$ and a length rate $h_1 \simeq 0.99$ (Figure 3 (a)) is characterized by a small radius increase. The second one has angles $(\frac{\pi}{4}, \frac{\pi}{8})$ and $h_1 \simeq 1.36$ (Figure 3 (b)).

Example 2. $(\varphi_1, \varphi_2) = (\frac{\pi}{3}, \frac{\pi}{4})$ and $\rho'(0) = 0$. The curve with $r^* = 1.25$ (Figure 4 (b)) is tighter than the curve with $r^* = 1$ (Figure 4 (a)).

Example 3. $r^* = 1.5$ and $(\varphi_1, \varphi_2) = (\frac{\pi}{5}, \frac{\pi}{4})$. We compare the curvature radius increase at the origin for a curve (Figure 5(a)) with $\rho'(0) = 0$ ($h_1 \simeq$ 1.71) and another one (Figure 5 (b)) with $\rho'(0) = -0.12\ \rho(0)$ ($h_1 \simeq 2.05$).

Example 4. We consider two curves with $r^* < 1$ and $\alpha^* = 1/r^*$: $r^* =$ 0.95, $(\varphi_1, \varphi_2) = (\frac{\pi}{4}, \frac{\pi}{5})$ and $r^* = 0.9$, $(\varphi_1, \varphi_2) = (\frac{\pi}{5}, \frac{\pi}{6})$. The parameters obtained by calculations (take the equality in (14)) are respectively $h_1 \simeq 1.44$, $\rho'(0) = -0.03\ \rho(0)$ and $h_1 \simeq 1.28$, $\rho'(0) = -0.06\ \rho(0)$ (Figure 6 (a)-(b)).

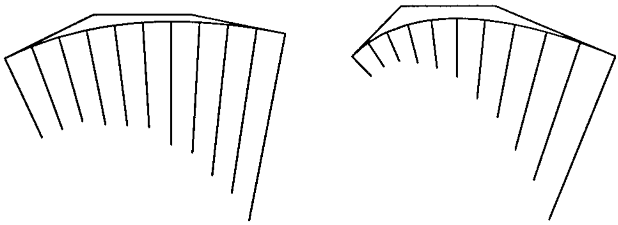

Fig. 3. **(a)** $r^* = 1$, $(\varphi_1, \varphi_2) = (\frac{\pi}{7}, \frac{\pi}{16})$, **(b)** $r^* = 1$, $(\varphi_1, \varphi_2) = (\frac{\pi}{4}, \frac{\pi}{8})$.

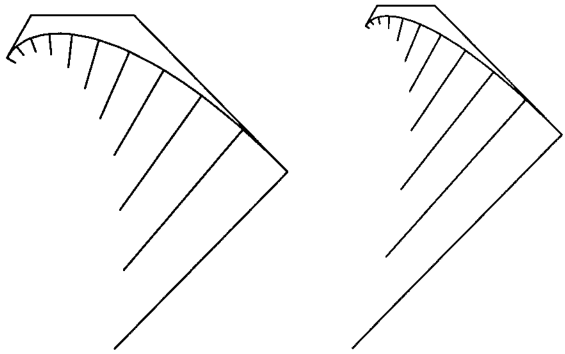

Fig. 4. (a) $r^* = 1$, $(\varphi_1, \varphi_2) = (\frac{\pi}{3}, \frac{\pi}{4})$, **(b)** $r^* = 1.25$, $(\varphi_1, \varphi_2) = (\frac{\pi}{3}, \frac{\pi}{4})$.

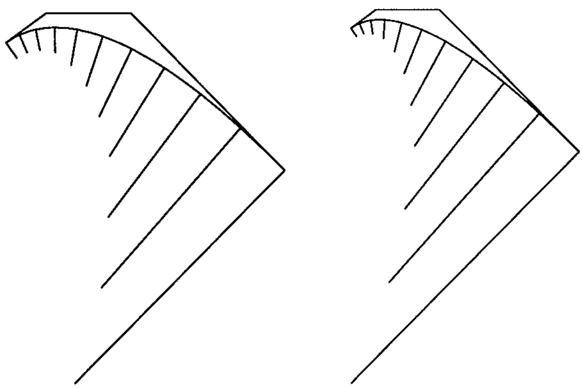

Fig. 5. $r^* = 1.5$ and $(\varphi_1, \varphi_2) = (\frac{\pi}{5}, \frac{\pi}{4})$ **(a)** $\rho'(0) = 0$, **(b)** $\rho'(0) = -0.12\rho(0)$.

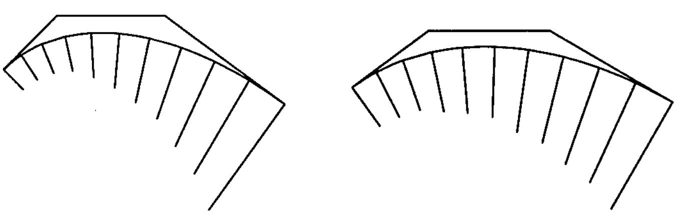

Fig. 6. (a) $r^* = 0.95$, $(\varphi_1, \varphi_2) = (\frac{\pi}{4}, \frac{\pi}{5})$, **(b)** $r^* = 0.9$, $(\varphi_1, \varphi_2) = (\frac{\pi}{5}, \frac{\pi}{6})$.

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